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ON STRONG P-POINTS

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ABSTRACT. This paper investigates the combinatorial property of ultrafilters that Mathias forcing relativized to them does not add dominating reals. We prove that the characterization due to Hrušák and Minami is equivalent to the strong P-point property. We also consistently construct a P-point that has no rapid Rudin-Keisler predecessor but that is not a strong P-point. These results answer questions of Canjar and Laflamme.

INTRODUCTION

In this paper we investigate conditions related to the question[1]:

Question. When does $\mathbb{M}_{\mathcal{U}}$, the Mathias forcing relativized to the ultrafilter $\mathcal{U}$, add a dominating real?

This question was first raised by M. Canjar in [2], who established the following necessary condition:

Theorem (Canjar)**.** *If $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real then $\mathcal{U}$ must be a P-point with no rapid Rudin-Keisler predecessor.*

He also proved that it is consistent with ZFC that there is an ultrafilter $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ does not add dominating reals and asked whether the above condition is sufficient. The topic was later studied by C. Laflamme in [5] where he introduced the notion of a strong P-point and noted without proof that this is also a necessary condition for $\mathbb{M}_{\mathcal{U}}$ not adding dominating reals. He asked whether it is sufficient and also whether it is equivalent to Canjar's condition. Ultrafilters $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ does not add dominating reals were also constructed and used by J. Brendle in [1].

Quite recently the topic was revisited by M. Hrušák and H. Minami in [3] who introduced a combinatorial condition on the ultrafilter $\mathcal{U}$ which they proved equivalent to $\mathbb{M}_{\mathcal{U}}$ not adding dominating reals. In this paper we show that this condition is equivalent to the ultrafilter being a strong P-point, and we consistently build a P-point without a rapid Rudin-Keisler predecessor

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[1]Unfamiliar concepts used in this introduction will be defined in Section 1.

which is, nevertheless, not a strong P-point. These results answer the questions of M. Canjar and C. Laflamme.

1. Preliminaries

We use standard notation, ω^ω denotes all functions from ω to ω, $[\omega]^\omega$ denotes all infinite subsets of ω while $[X]^{<\omega}$ denotes all finite subsets of X. We write $A \subseteq^* B$ if $|A \setminus B| < \omega$ and $f \leq^* g$ if $|\{n : f(n) > g(n)\}| < \omega$. The cardinal number $\mathfrak{b}$ denotes the least cardinality of an unbounded subset of $(\omega^\omega, \leq^*)$ and $\mathfrak{d}$ denotes the least cardinality of a dominating (cofinal) subset of $(\omega^\omega, \leq^*)$. $\chi(\mathcal{U})$ denotes the *character* of $\mathcal{U}$, i.e. the least cardinality of a basis of the (ultra)filter $\mathcal{U}$.

1.1. **Definition** ([10]). A nonprincipal ultrafilter $\mathcal{U}$ is a *P-point* if for any sequence $\langle X_n : n < \omega \rangle \subseteq \mathcal{U}$ there is an $X \in \mathcal{U}$ such that $(\forall n < \omega)(X \subseteq^* X_n)$.

1.2. **Definition** ([9]). An ultrafilter $\mathcal{U}$ is *rapid* if the family $\{e_X : X \in \mathcal{U}\}$ of increasing enumerations of sets in $\mathcal{U}$ is a dominating family of functions in $(\omega^\omega, \leq^*)$.

1.3. **Definition.** Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on ω.

(i) (Rudin-Keisler ordering, [4]) $\mathcal{U} \leq_{RK} \mathcal{V}$ if there is a function $f : \omega \to \omega$ such that $\mathcal{U} = f_*(\mathcal{V}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{V}\}$. In this situation we also say that $\mathcal{U}$ is an RK-predecessor of $\mathcal{V}$.

(ii) (Rudin-Blass ordering, [6]) $\mathcal{U} \leq_{RB} \mathcal{V}$ if $\mathcal{U} \leq_{RK} \mathcal{V}$ and the function witnessing this can be chosen to be finite-to-one. As above we say that $\mathcal{U}$ is an RB-predecessor of $\mathcal{V}$.

1.4. **Definition** ([8]). *Mathias forcing* is the partial order where conditions are pairs (a, X) with $a \in [\omega]^{<\omega}$ and $X \in [\omega]^\omega$ ordered as $(a, X) \leq (b, Y)$ if $b \sqsubseteq a$, $X \subseteq Y$ and $a \setminus b \subseteq Y$. Given an ultrafilter $\mathcal{U}$, *relativized Mathias forcing* $\mathbb{M}_\mathcal{U}$ is the subset of Mathias forcing consisting of conditions whose second coordinate is in $\mathcal{U}$.

1.5. **Remark.** Mathias forcing can be written as an iteration $\mathbb{M} = \mathcal{P}(\omega)/\text{fin} * \mathbb{M}_{\dot{G}}$, where $\dot{G}$ is a name for the generic ultrafilter added by the first forcing. It is easy to verify that the generic real for relativized Mathias forcing $\mathbb{M}_\mathcal{U}$, which is the union of the first coordinates of conditions in the generic filter, is a pseudointersection of $\mathcal{U}$.

2. Characterization of Canjar ultrafilters

2.1. **Definition.** A *Canjar ultrafilter* is an ultrafilter on ω such that $\mathbb{M}_\mathcal{U}$ does not add dominating reals.

The following observation will motivate the definition of a strong P-point.

2.2. **Observation.** *An ultrafilter $\mathcal{U}$ is a P-point if and only if for any descending sequence of sets $\langle X_n : n < \omega \rangle$ from $\mathcal{U}$ there is an interval partition $\langle I_n : n < \omega \rangle$ of ω such that*

$$X = \bigcup_{n<\omega} (I_n \cap X_n) \in \mathcal{U}.$$

Note that X will always be a pseudointersection of the X_n's, and the larger the intervals are, the larger it will be.

Generalizing the above observation, C. Laflamme introduced:

2.3. **Definition** ([5])**.** An ultrafilter is a *strong P-point* if for any sequence $\langle \mathcal{C}_n : n < \omega\rangle$ of compact subsets of $\mathcal{U}$ (considering $\mathcal{U}$ as a subset of 2^ω with the product topology), there is an interval partition $\langle I_n : n < \omega\rangle$ such that for each choice of $X_n \in \mathcal{C}_n$ we have

$$X = \bigcup_{n<\omega} (I_n \cap X_n) \in \mathcal{U}.$$

It is easy to see that a strong P-point cannot be rapid (for example consider $\mathcal{C}_n = \{X : |\omega \setminus X| \leq n\}$) and in [5, Lemma 6.8] it is proved that strong P-points are preserved when passing to RK-predecessors. So we have:

2.4. **Fact** (Laflamme)**.** *A strong P-point is a P-point with no rapid RK-predecessors.*

The following notion was probably first considered implicitly by S. M. Sirota ([11]) and explicitly by A. Louveau ([7]) in the construction of an extremally disconnected topological group:

2.5. **Notation.** Given a filter $\mathcal{F}$ on ω we define $\mathcal{F}^{<\omega}$ to be the filter on $[\omega]^{<\omega} \setminus \{\emptyset\}$ generated by $\{[F]^{<\omega} \setminus \{\emptyset\} : F \in \mathcal{F}\}$. Note that $[\mathcal{F}]^{<\omega}$ is a filter on $[\omega]^{<\omega}$ and it is *not* an ultrafilter even if $\mathcal{F}$ is.

2.6. **Definition.** A filter $\mathcal{F}$ on a countable set S is a P^+-filter if, for any descending sequence $\langle X_n : n < \omega\rangle \subseteq \mathcal{F}^+$, there is an $X \in \mathcal{F}^+$ such that $X \subseteq^* X_n$ for all n, where $\mathcal{F}^+ = \{X \subseteq S : S \setminus X \notin \mathcal{F}\}$.

2.7. **Lemma.** *If $\mathcal{U}$ is an ultrafilter then $A \subseteq [\omega]^{<\omega}$ is $\mathcal{U}^{<\omega}$-positive if and only if each set X such that every element $a \in A$ has nonempty intersection with X is in $\mathcal{U}$.*

Proof. Suppose A is positive and X hits each element of A. Pick $Y \in \mathcal{U}$. Then $[Y]^{<\omega} \cap A \neq \emptyset$ so $Y \cap X \neq \emptyset$. Since $\mathcal{U}$ is an ultrafilter, $X \in \mathcal{U}$. On the other hand if A is not positive there is some $Y \in \mathcal{U}$ with $[Y]^{<\omega} \cap A = \emptyset$. Then $X = \omega \setminus Y$ hits every element of A. □

2.8. **Theorem** ([3])**.** *$\mathbb{M}_\mathcal{U}$ does not add a dominating real if and only if $\mathcal{U}^{<\omega}$ is a P^+-filter.*

We extend this result by proving the following theorem:

2.9. **Theorem.** *For an ultrafilter $\mathcal{U}$ the following are equivalent*

(i) *$\mathcal{U}$ is Canjar, i.e. $\mathbb{M}_\mathcal{U}$ does not add a dominating real.*
(ii) *$\mathcal{U}^{<\omega}$ is a P^+-filter.*
(iii) *$\mathcal{U}$ is a strong P-point*

The implication from (i) to (iii) was already known to C. Laflamme but, as far as we know, was never published.

Proof. (i) being equivalent with (ii) is proved in [3]. To make the paper self-contained we include the proof.

(ii)⇒(i): Assume $\mathcal{U}^{<\omega}$ is a P$^+$-filter and suppose, aiming towards a contradiction, that $\mathbb{M}_\mathcal{U}$ adds a dominating real. Let $\dot{g}$ be a name for it. For each $f \in \omega^\omega$ there is an $n_f < \omega$ and $(t_f, F_f) \in \mathbb{M}_\mathcal{U}$ such that

$$(t_f, F_f) \Vdash (\forall k \geq n_f)(f(k) \leq \dot{g}(k)).$$

Since $\mathfrak{b} > \omega$, we can fix $n < \omega$ and $t \in [\omega]^{<\omega}$ such that the family of functions $\mathcal{F} = \{f \in \omega^\omega : n_f = n \ \& \ t_f = t\}$ is a dominating family. For $k < \omega$ let

$$X'_k = \{s \in [\omega \setminus t]^{<\omega} : (\exists F \in \mathcal{U}, m \geq k, i < \omega)((t \cup s, F) \Vdash \dot{g}(m) = i)\}.$$

Clearly X'_k is $\mathcal{U}^{<\omega}$-positive and the sets decrease as k increases. Define $Y = \bigcap_{k<\omega} X'_k$ and let $X_k = X'_k \setminus Y$. Notice that the sets X_k are still decreasing and, if we can show that Y is not $\mathcal{U}^{<\omega}$-positive, then they will also be positive.

Claim $Y \notin (\mathcal{U}^{<\omega})^+$. Suppose otherwise and for $s \in Y$ let $f_s : A_s \to \omega$ be a maximal (w.r.t. inclusion) function such that for each $m \in A_s$ there is $F^s_m \in \mathcal{U}$ such that $(t \cup s, F^s_m) \Vdash \dot{g}(m) = f_s(m)$. Note that each A_s is infinite. Choose $f \in \mathcal{F}$ eventually dominating $\{f_s : s \in Y\}$. Pick $F \in \mathcal{U}$ such that $(t, F) \Vdash (\forall m > n)(f(m) \leq \dot{g}(m))$. Since Y is positive, there must be some $s \in Y \cap [F]^{<\omega}$. Finally pick $m > n$ such that $m \in A_s$ and $f_s(m) < f(m)$ (this is possible since A_s is infinite and f eventually dominates f_s). But then $(t \cup s, F \cap F^s_m) \Vdash \dot{g}(m) = f_s(m) < f(m) \leq \dot{g}(m)$ — a contradiction. This completes the verification of the claim.

Since $\mathcal{U}^{<\omega}$ is a P$^+$-filter by assumption, there must be a $\mathcal{U}^{<\omega}$-positive set $X \subseteq X_0$ which is a pseudointersection of the X_k's. Define

$$f(k) = \max\{i + 1 : (\exists s \in X \setminus X_{k+1}, F \in \mathcal{U})((t \cup s, F) \Vdash \dot{g}(k) = i)\} \cup \{0\}.$$

Since the family $\mathcal{F}$ was a dominating family, choose $h \in \mathcal{F}$ dominating f above some $k_0 < \omega$ with $n < k_0$. Since X is $\mathcal{U}^{<\omega}$-positive and $X \subseteq^* X_{k_0}$, we may find $s \in X \cap X_{k_0} \cap [F_h]^{<\omega}$. Let k be maximal such that $s \in X_k$. Then $k \geq k_0$. By the definition of the X_k's and f, there is $F \in \mathcal{U}$ and $i < f(k)$ such that $(t \cup s, F) \Vdash \dot{g}(k) = i$. But this contradicts the fact that $(t \cup s, F \cap F_h) \leq (t, F_h)$ forces $f(k) \leq \dot{g}(k)$.

(i)⇒(ii): Assume $\mathcal{U}^{<\omega}$ is not a P$^+$-filter. We shall show that $\mathcal{U}$ is not Canjar. Let $\langle X_n : n < \omega \rangle$ be a descending sequence of $\mathcal{U}^{<\omega}$-positive sets with no positive pseudointersection. Work in the extension by $\mathbb{M}_\mathcal{U}$ and let $F_g \subseteq \omega$ be the generic real. Notice that $[F_g \setminus n]^{<\omega} \cap X_n \neq \emptyset$. Otherwise there would be some condition (s, A) forcing $[F_g \setminus n]^{<\omega} \cap X_n = \emptyset$. However, since X_n positive with respect to $\mathcal{U}^{<\omega}$, there would be $t \in [A \setminus n]^{<\omega} \cap X_n$, and $(s \cup t, A) \Vdash t \in [F_g \setminus n]^{<\omega} \cap X_n$. This would contradict the choice of (s, A). So $[F_g \setminus n]^{<\omega} \cap X_n \neq \emptyset$ and we can recursively pick $x_n \in [F_g \setminus n]^{<\omega} \cap X_n$. Let $f(n) = \max x_n + 1$ and notice that $x_n \in [f(n)]^{<\omega} \cap X_n$. Suppose some strictly increasing h is not dominated by f and let $X = \bigcup_{n<\omega} [h(n)]^{<\omega} \cap X_n$. Clearly X is a pseudointersection of the X_n's. Since h is not dominated by f, X contains infinitely many x_i's and it follows that it is positive: Suppose $F \in \mathcal{U}$. We will show $[F]^{<\omega} \cap X \neq \emptyset$. Find $n < \omega$ such that $F_g \setminus n \subseteq F$. Then we can pick $m > n$ such that $x_m \in X$ and $x_m \subseteq F_g \setminus m \subseteq F_g \setminus n \subseteq F$. This shows that X is positive. So h cannot be in the ground model.

(ii)⇒(iii): Suppose $\mathcal{U}^{<\omega}$ is a P$^+$-filter but $\mathcal{U}$ is not a strong P-point. Let $\mathcal{C}_n$ witness the latter. We may assume that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ and that $\mathcal{C}_n$ is closed under intersections of up to $n+1$ elements, i.e. $(\forall \mathcal{C} \in [\mathcal{C}_n]^{\leq n+1})(\bigcap \mathcal{C} \in \mathcal{C}_n)$. Let

$$A_n = \{a \in [\omega]^{<\omega} : (\forall X \in \mathcal{C}_n)(a \cap X \neq \emptyset)\}.$$

Notice that $A_{n+1} \subseteq A_n$. In addition $A_n \in (\mathcal{U}^{<\omega})^+$. To show this, choose $F \in \mathcal{U}$ and check that $\{X \cap F : X \in \mathcal{C}_n\}$ is a compact set not containing $\emptyset$. In particular there is an $a \in [F]^{<\omega}$ such that $a \cap X \neq \emptyset$ for each $X \in \mathcal{C}_n$, so $a \in A_n \cap [F]^{<\omega}$. Now let A be a $\mathcal{U}^{<\omega}$-positive pseudointersection of the A_n's. We will deduce a contradiction. Let

$$g(n) = \min\{k : a \in A \setminus A_n \to a \subseteq k\}.$$

Enlarging $g(n)$, if necessary, we may assume it is increasing. By our assumption on the $\mathcal{C}_n$'s, there are X_n's with $X_n \in \mathcal{C}_n$ such that

$$\bigcup_{n<\omega} \big(X_n \cap [g(n), g(n+1))\big) \notin \mathcal{U}.$$

Define $Y_n = \bigcap_{i \leq n} X_n$ and notice that $Y_n \in \mathcal{C}_n$ since the sequence of $\mathcal{C}_n$'s is increasing and $\mathcal{C}_n$ is closed under intersections of at most $n+1$ elements. Moreover we have

$$Y = \bigcup_{n<\omega} \big(Y_n \cap [0, g(n+1))\big) \subseteq \bigcup_{n<\omega} \big(X_n \cap [g(n), g(n+1))\big) \notin \mathcal{U}.$$

Since A is positive Lemma 2.7 will give the desired contradiction if we show that Y hits each $a \in A$. Pick $a \in A$ and let

$$k = \max\{n : a \cap [g(n), g(n+1)) \neq \emptyset\}$$

Notice that $a \subseteq g(k+1)$ and, by the definition of g, $a \in A_k$. Hence $a \cap Y_k \neq \emptyset$ so $a \cap [0, g(k+1)) \cap Y_k \neq \emptyset$ so $a \cap Y \neq \emptyset$ and we are done.

(iii)⇒(ii): Suppose on the other hand that $\mathcal{U}$ is a strong P-point and that $\langle A_n : n < \omega \rangle$ is a descending sequence of $\mathcal{U}^{<\omega}$-positive sets. We shall find a $\mathcal{U}^{<\omega}$-positive pseudointersection. Let

$$\mathcal{C}_n = \{X : (\forall a \in A_n)(a \cap X \neq \emptyset)\}.$$

Then $\mathcal{C}_n \subseteq \mathcal{U}$ by Lemma 2.7. Moreover $\mathcal{C}_n$ is closed (it is an intersection of clopen sets). Since $\mathcal{U}$ is a strong P-point, there is an interval partition $\langle I_n : n < \omega \rangle$ of ω satisfying the condition in the definition of a strong P-point. Let

$$A = \bigcup_{n<\omega} \big(A_n \cap \mathcal{P}(I_n)\big).$$

Since the A_n's were decreasing, A will be a pseudointersection of them. We have to show that it is positive. Pick $F \in \mathcal{U}$. We need to show that there is $n < \omega$ such that $[F]^{<\omega} \cap A_n \cap \mathcal{P}(I_n) \neq \emptyset$. Suppose this is not so. Then let $X_n = (\omega \setminus I_n) \cup (I_n \setminus F)$ and notice that $\bigcup_{n<\omega}(X_n \cap I_n) = \omega \setminus F \notin \mathcal{U}$. We will show that each $X_n \in \mathcal{C}_n$ which will contradict the choice of the interval partition, thus finishing the proof. But given some $a \in A_n$ either $a \notin \mathcal{P}(I_n)$ and then $a \cap X_n \neq \emptyset$ trivially, or $a \in \mathcal{P}(I_n)$ but then $a \notin [F]^{<\omega}$ so $a \cap X_n \neq \emptyset$ also. □

3. A consistent example

The aim of this section is to construct a P-point which has no rapid RK-predecessor and which, at the same time, is not a strong P-point. Of course, the construction will require a hypothesis beyond ZFC, since S. Shelah [13] has shown that ZFC does not prove the existence of P-points. The continuum hypothesis is more than adequate for our construction; in fact we use the weaker hypothesis that $cov(\mathcal{M}) = \mathfrak{c}$.

We will need the following characterization of rapid ultrafilters due to P. Vojtáš:

3.1. **Definition.** An ideal I on ω is a *tall summable ideal* if there is a function $g : \omega \to \mathbb{R}_0^+$ which tends to zero and

$$I = \left\{ A \subseteq \omega : \mu_g(A) = \sum_{n \in A} g(n) < \infty \right\} = I_g$$

3.2. **Theorem** ([12]). *An ultrafilter is rapid if and only if it meets each tall summable ideal.*

3.3. **Theorem.** *Assume $cov(\mathcal{M}) = \mathfrak{c}$ There is an ultrafilter which is a P-point with no RK-predecessors but is not a strong P-point.*

Note. $cov(\mathcal{M})$ is the minimal cardinality of a family of meager sets covering 2^ω. We shall use the fact that $cov(\mathcal{M}) = \mathfrak{c}$ is equivalent to $MA(ctble)$, Martin's axiom for countable partial orders.

Proof of the theorem. The construction is a classical induction proof where at each step we kill potential witnesses to strong P-pointness and to rapid filters below while guaranteeing that the constructed ultrafilter will be a P-point. Note that since the resulting ultrafilter will be a P-point, we need only check RB-predecessors because of

3.4. **Fact.** *Any nonprincipal RK-predecessor of a P-point is an RB-predecessor of it.*

Let $\langle A_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of $\mathcal{P}(\omega)$, $\langle \langle I_n^\alpha : n < \omega \rangle, \alpha < \mathfrak{c} \rangle$ be an enumeration of all interval partitions, $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ an enumeration of all finite-to-one functions from ω to ω, and, finally, $\langle \langle A_n^\alpha : n < \omega \rangle, \alpha < \mathfrak{c} \rangle$ an enumeration of all countable sequences of subsets of ω with each sequence appearing cofinally often. Let $\mu(A) = \sum_{n \in A} 1/n$ and

$$\mathcal{C}_n = \{ X \subseteq \omega : \mu(\omega \setminus X) \leq n + 1 \}.$$

These $\mathcal{C}_n$'s will witness the failure of the strong P-point property.

By recursion we shall construct filters $\langle \mathcal{U}_\alpha : \alpha < \mathfrak{c} \rangle$ such that the following conditions are met:

(i) $\chi(\mathcal{U}_\alpha) \leq \alpha + \omega$,
(ii) $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ for $\alpha \leq \beta < \mathfrak{c}$,
(iii) each $X \in \mathcal{U}$ has $\mu(X) = \infty$,
(iv) $A_\alpha \in \mathcal{U}_{\alpha+1}$ or $\omega \setminus A_\alpha \in \mathcal{U}_{\alpha+1}$,
(v) There are $A \in \mathcal{U}_{\alpha+1}$ and $X_n \in \mathcal{C}_n$ such that $A \cap \bigcup_{n<\omega} (X_n \cap I_n^\alpha) = \emptyset$
(vi) There is a $g : \omega \to \mathbb{R}_0^+$ tending to zero and there is an $A \in \mathcal{U}_{\alpha+1}$ such that $(\forall B \in I_g)(\mu(A \cap f_\alpha^{-1}[B]) < \infty)$

(vii) If $\langle A_n^\alpha : n < \omega\rangle \subseteq \mathcal{U}_\alpha$ then there is an $A \in \mathcal{U}_{\alpha+1}$ which is a pseudointersection of the sequence $\langle A_n^\alpha : n < \omega\rangle$.

It is easy to see that if we are able to build such a sequence of filters and we let $\mathcal{U} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{U}_\alpha$ then $\mathcal{U}$ will be as required: (iv) shows that $\mathcal{U}$ will be an ultrafilter, (v) shows that the $\mathcal{C}_n$'s witness $\mathcal{U}$ is not a strong P-point and (vii) shows that $\mathcal{U}$ is a P-point. We will show that, because of (vi), $\mathcal{U}$ will have no rapid RB-predecessors.

Suppose f is a finite-to-one function. Then $f = f_\alpha$ for some $\alpha < \mathfrak{c}$ and we know that there are g and A as in (vi). Then, given $B \in I_g$, $\mu(f^{-1}[B] \cap A) < \infty$ so, by (iii) and since $\mathcal{U}$ is an ultrafilter, there is $X \in \mathcal{U}$ disjoint from $f^{-1}[B]$. So $B \notin f_*(\mathcal{U})$. This shows that $f_*(\mathcal{U}) \cap I_g$ is empty and, by Theorem 3.2, $f_*(\mathcal{U})$ is not rapid.

We will now show that the recursive construction can be carried out. Conditions (i), (ii) and (iii) will keep the induction going. At limit stages take unions and all conditions will be satisfied. Consider now the successor stages. Assume $\mathcal{U}_\alpha$ is constructed.

To guarantee (iv), note that by (iii) $\{A : \mu(\omega \setminus A) < \infty\} \cup \mathcal{U}_\alpha$ generates a filter which can be extended by either A_α or $A_{\alpha+1}$ to $\mathcal{U}'_\alpha$ satisfying (i-iv).

To get (v): Consider the forcing notion

$$\mathbb{C}_0 = \{a \in [\omega]^{<\omega}, (\forall n < \omega)(\mu(a \cap I_n^\alpha) \leq n + 1)\}$$

ordered by reverse inclusion. By (i) we can fix some basis $\mathcal{B}$ of $\mathcal{U}'_\alpha$ of size $< \mathfrak{c}$ and let $D_X^k = \{a \in \mathbb{C}_0 : \mu(a \cap X) \geq k\}$ for $X \in \mathcal{B}$. Clearly each D_X^k is dense (since each $X \in \mathcal{B}$ has infinite measure) so as $cov(\mathcal{M}) = \mathfrak{c}$ there is an $A \subseteq \omega$ generic for these sets. If we let $X_n = \omega \setminus (I_n^\alpha \cap A)$ then $X_n \in \mathcal{C}_n$ and $A \cap \bigcup_{n<\omega}(I_n^\alpha \cap X_n) = \emptyset$. Moreover, since A was generic the filter $\mathcal{U}''_\alpha$ generated by $\mathcal{U}'_\alpha$ and A will satisfy (i-v).

To get (vi): Let $b_n = f_\alpha^{-1}[\{n\}]$. For $a \in [\omega]^{<\omega}$ let

$$s(a) = max\{n : a \cap b_n \neq \emptyset\}$$

Consider the forcing $\mathbb{C}_1 = \{(a, q) : a \in [\omega]^{<\omega}, q \in \mathbb{Q}\}$ ordered as follows: $(a, p) \leq (c, q)$ if $a \supseteq c$, $(\forall n \leq s(c))(a \cap b_n = c \cap b_n)$, $p \leq q$ and finally $(\forall n \geq s(c))(\mu(a \cap b_n) \leq q)$. Again for a fixed basis $\mathcal{B}$ of $\mathcal{U}''_\alpha$ of size $< \mathfrak{c}$ and $X \in \mathcal{B}$ let $D_X^k = \{(a, q) \in \mathbb{C}_1 : \mu(a \cap X) \geq k, q \in \mathbb{Q}\}$ and notice that these are dense sets. So the generic A can be added to $\mathcal{U}''_\alpha$ to get $\mathcal{U}'''_\alpha$ satisfying (iii), and hence(i-v). Moreover for $q \in \mathbb{Q}$ the set $D_q = \{(a, p) \in \mathbb{C}_1 : p \leq q\}$ is dense which shows that if we let $g(n) = \mu(A \cap b_n)$ then $g(n) \to 0$ so $\mathcal{U}'''_\alpha$ satisfies (i-vi).

Finally for (vii), suppose $A_n^\alpha \in \mathcal{U}_\alpha$ for every $n < \omega$ and consider the set $\mathbb{C}_2 = \{(a, K) : a \in [\omega]^{<\omega}, K \in [\omega]^{<\omega}\}$ ordered as follows $(a, K) \leq (b, L)$ if $a \supseteq b$, $K \supseteq L$, and $a \setminus b \in \bigcap_{n\in L} A_n^\alpha$. Again fix a basis $\mathcal{B}$ for $\mathcal{U}''''_\alpha$ of size $< \mathfrak{c}$. Then the set $D_X^m = \{(a, K) \in \mathbb{C}_2 : \mu(a \cap X) \geq m\}$ is dense for each $m \in \omega, X \in \mathcal{B}$ since $\mu(A_n^\alpha) = \infty$. This guarantees that the generic set can be added to $\mathcal{U}''''_\alpha$ to get $\mathcal{U}_{\alpha+1}$ satisfying (iii), and hence (i-vi). Also, for any $n < \omega$ the set $D_n = \{(a, K) \in \mathbb{C}_2 : n \in K\}$ is dense and this shows that the generic set will be a pseudointersection of the A_n^α's, so $\mathcal{U}_{\alpha+1}$ satisfies (i-vii). $\square$

3.5. **Remark.** Theorem 3.3 is in some sense optimal, i.e. $\mathfrak{d} = \mathfrak{c}$ is not sufficient. To see this notice that in the Miller model $\mathfrak{d} = \mathfrak{c}$, while all P-points have character $< \mathfrak{d}$ and so must satisfy the combinatorial condition from [3] (any filter of character $< \mathfrak{d}$ is a P^+-filter; also $\mathbb{M}_{\mathcal{U}}$ has a dense subset of size $< \mathfrak{d}$ and therefore cannot add a dominating function). In this model all P-points are in fact strong P-points.

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