Lonely points revisited

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Abstract

We use the space $\mathcal{G}_\omega$ from [DGS88] to construct a new topological type in ω^*. This answers a question of P. Simon posed in [Ver08].

1 Introduction

1.1 Definition. *A* topological type *in a space* X *is a subset* $T \subseteq X$ *which is invariant under homeomorphisms.*

An example of a topological type are the discrete points in a space X. Another more interesting type is given in the following definition. The first part is due to W. Rudin ([Rud56]) the second to K. Kunen ([Kun78]).

1.2 Definition (Rudin, Kunen)**.** *A point* $x \in X$ *is a* P-point *if the countable intersection of neibourghoods of* x *is again a neibourghood of* x*. It is a* weak P-point *if it is not a limit point of a countable subspace of* X.

Clearly any isolated point is a P-point, and a P-point is a weak P-point. However none of the implications can be reversed.

If a space contains two distinct topological types, then it is not homogeneous. The motivation for finding topological types in ω^* was given by the following surprising result of Z. Frolík ([Fro67a],[Fro67b]):

1.3 Theorem (Frolík)**.** ω^* *is not homogeneous.*

His proof was a clever combinatorial argument but it gave no intrinsically topological reason for the non-homogeneity of ω^*. This motivated the question whether one can find a "topologically defined" topological type — an "honest" proof of nonhomogeneity. Under CH, this was answered already by W. Rudin in [Rud56] where he proved that P-points exist in ω^*. However in ZFC the question remained open for some twenty years.

In his seminal paper [Kun78], K. Kunen proved in ZFC that ω^* contains a weak P-point:

1.4 Theorem (Kunen)**.** ω^* *contains a weak P-point.*

Since it obviously contains non weak P-points, this is an "honest" proof of nonhomogeneity. In [vMill82], J. van Mill had exploited the techniques of K. Kunen to prove, in ZFC, the existence of sixteen distinct topological types in ω^*! One of the types he introduced is given in the following theorem:

1.5 Theorem (van Mill)**.** *There is a point* $p \in \omega^*$ *which is a limit point of a countable discrete set and the countable sets whose limit point it is form a filter.*

Proof. (Idea) Use Kunen's result to construct a weak P-point $p \in \omega^* \subseteq \beta\omega$. Now use theorem 2.5 (due to P. Simon) to embed $\beta\omega$ into ω^* via f as a weak P-set. Then $f(p)$ will be as required, since p clearly has the property in $\beta\omega$ and the embedding does not destroy it as $f[\beta\omega]$ is a weak P-set. □

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This motivated P. Simon to define the following notion, which we have called a lonely point in [Ver08]. We want essentially the same type of point as in the above theorem only replacing the countable discrete set whose limit point it is by a crowded set:

1.6 DEFINITION. *A point* $p \in X$ *is a* lonely point *provided:*

(i) p *is* ω*-discretely untouchable, i.e. not a limit point of a countable discrete set,*

(ii) p *is a limit point of a countable crowded (i.e. without isolated points) set and*

(iii) The countable sets whose limit point p *is form a filter.*

In the paper we were able to show that they exist in some open dense subspace of ω^*. Here we prove that they actually exist in ω^*:

1.7 THEOREM. ω^* *contains a lonely point.*

The idea is to construct a countable, OHI, ED space X with an $\aleph_0$-bounded remainder and then embed it as a weak P-set into ω^*. Any point of X will then be a lonely point of βX and, since βX will be a weak P-set in ω^*, also a lonely point of ω^*.

2 Basic definitions and theorems

2.1 DEFINITION (Kunen)**.** $F \subseteq X$ *is a weak P-set of* X *if any countable* $D \subseteq X$ *disjoint from* F *has closure disjoint from* F.

2.2 OBSERVATION. *If* $F \subseteq X$ *is a weak P-set of* X *and* $x \in F$ *is a lonely point of* F *then it is also a lonely point of* X.

2.3 DEFINITION. *A space* X *is* extremally disconnected *(or ED for short) if the closure of any open set is open.*

The following is standard, see e.g. [Eng]:

2.4 THEOREM. *If* X *is ED then so is* βX

We shall also need the following theorem of P. Simon (see [Sim85]):

2.5 THEOREM (Simon)**.** *The Čech-Stone compactification of any* T_3 *ED space of weight* $\leq 2^{\aleph_0}$ *can be embedded into* ω^* *as a closed weak* P*-set.*

3 Irresolvable spaces

In this section, unless otherwise stated, we assume all spaces to be crowded (i.e. without isolated points). The following definitions were introduced in [vD93]:

3.1 DEFINITION (van Douwen)**.** *A crowded space* X *is* perfectly disconnected *if no point of* X *is a limit point of two disjoint subsets of* X*. It is irresolvable, if it contains no disjoint dense sets. It is open-hereditarily-irresolvable (OHI for short), provided each open subspace is irresolvable. A crowded space is* maximal regular *if each finer topology either contains an isolated point or is not regular.*

Irresolvable spaces were constructed by E. Hewitt ([Hew43]) and independently by M. Katětov ([Kat47]). They were extensively studied in [vD93] where the following theorems may be found:

3.2 THEOREM ([vD93],1.7,1.11)**.** *Maximal regular spaces are zerodimensional, ED and OHI.*

3.3 Theorem ([vD93],1.4,1.6). *If A, B are disjoint crowded subspaces of a maximal regular space, then $\overline{A}$ and $\overline{B}$ are disjoint.*

3.4 Theorem ([vD93],2.2). *If X is ED and OHI and each nowhere dense subset of X is closed then X is perfectly disconnected.*

The following theorem is not explicitly stated in van Douwen's paper, but its proof is essentially given in his Lemma 3.2 and Example 3.3.

3.5 Theorem (van Douwen). *Any countable maximal regular space X contains an open perfectly disconnected subspace.*

Proof. For each $Z \subseteq X$ let

$$A_Z = \{x \in Z : x \text{ is a limit point of a relatively discrete subset of } Z\}$$

Claim $A_Z \neq Z$ for each open subset Z of X.

Assume otherwise. Enumerate X as $\langle x_n : n < \omega\rangle$. By induction construct pairwise disjoint, relatively discrete sets $\langle D_n : n < \omega\rangle$ such that:

(i) $\bigcup_{i<n} D_i \subseteq \overline{D_n}$ for all $n < \omega$ and

[(ii) $x_n \in \overline{D_n}$ for $n < \omega$.

This will lead to a contradiction with the irresolvability of Z (by theorem 3.2, X is OHI, so Z is irresolvable) since $\bigcup_{n<\omega} D_{2n}$ and $\bigcup_{n<\omega} D_{2n+1}$ would then be disjoint dense subsets of Z. To see that the construction can be carried out let $D_0 = \{x_0\}$ and assume we have constructed D_i for $i \leq n$. Let $Y = D_n \cup X \setminus \overline{D_n}$. Since D_n is relatively discrete, Y is open. Since Z is regular and D_n is countable and relatively discrete, there is a pairwise disjoint collection of open sets $\{U_x : x \in D_n\}$ such that $x \in U_x \subseteq Y$. Since we assumed $A_Z = Z$ we can choose for each $x \in D_n$ a relatively discrete set D_x such that $D_x \subseteq U_x$ and $x \in \overline{D_x} \setminus D_x$. Let $D'_{n+1} = \bigcup_{x\in D_n} D_x$. If x_{n+1} is a limit point of D'_{n+1} let $D_{n+1} = D'_{n+1}$ otherwise let $D_{n+1} = D'_{n+1} \cup \{x_{n+1}\}$. Then D_{n+1} is as required.

Claim $\operatorname{int} A_X = \emptyset$.

For any clopen U, $A_X \cap U = A_U$. Since X is regular and countable, it is zerodimensional. Suppose U is clopen and $U \subseteq A_X$. By the previous claim $U \setminus A_U \neq \emptyset$ but then $U \setminus A_X \neq \emptyset$ a contradiction.

Claim A_X is nowhere dense.

Take any open $U \subseteq X$. Then $U \setminus A_X$ is dense in U_X, since $\operatorname{int} A_X = \emptyset$. Since X is OHI (by theorem 3.2), U is irresolvable so A_X cannot be dense in U so $U \not\subseteq \overline{A_X}$. Thus $\operatorname{int} \overline{A_X} = \emptyset$.

Claim If $A \subseteq X$ is nowhere dense then there is a discrete $D \subseteq A$ dense in A.

Let $D = \{x \in A : x \text{ is isolated in } A\}$. Since X is regular and countable D is relatively discrete. Since A is nowhere dense, D is discrete. Let $E = A \setminus \overline{D}$. Then E has no isolated points. Also $X \setminus E$ has no isolated points. By theorem 3.3 E must be open which contradicts that A is nowhere dense.

Let

$$\vartheta = \{x \in X : x \text{ is not a limit point of a nowheredense subset of } X\}$$

By the previous claim (and by the fact that each discrete subset of X is nowheredense)

$$\vartheta = \{x \in X : x \text{ is not a limit point of a discrete set}\}$$

Then $X \setminus \vartheta \subseteq A_X$ so $X \setminus \vartheta$ is nowhere dense, so $\operatorname{int} \vartheta$ is nonempty. We finally show that $\operatorname{int} \vartheta$ is perfectly disconnected. By the definition of ϑ any nowhere dense subset of $\operatorname{int} \vartheta$ is closed. Now it remains to apply theorem 3.4 remembering that by theorem 3.2 $\operatorname{int} \vartheta$ is ED and OHI (any open subspace of a maximal regular space is maximal regular). □

4 Proof of the main theorem

The following definition and theorem is taken from [DGS88]:

4.1 Definition. *Let $p \in \omega^*$ be a weak P-point. The space $\mathcal{G}_\omega$ is the space $\omega^{<\omega}$ of all finite sequences of natural numbers with $G \subseteq \omega^{<\omega}$ being open precisely when for each $\sigma \in G$ the set $\{n : \sigma^\frown n \in G\}$ is in p.*

4.2 Theorem (Dow, Gubbi, Szymanski)**.** *The remainder of $\mathcal{G}_\omega$ is $\aleph_0$-bounded. Moreover $\mathcal{G}_\omega$ is a T_2, zerodimensional, ED space.*

To make this paper selfcontained, we include a proof of the above theorem.

Proof. It is clear that the space is T_2: Given $\sigma, \tau \in \mathcal{G}_\omega$ if $\sigma \perp \tau$ then $G_\sigma = \{s \in \mathcal{G}_\omega : \sigma \subseteq s\}$ and $G_\tau = \{s \in \mathcal{G}_\omega : \tau \subseteq s\}$ are disjoint open sets separating σ from τ. If $\sigma \subseteq \tau$ let $G_\sigma = \{s \in \mathcal{G}_\omega : \tau \not\subseteq s\}$ and G_τ as before. Again we get two disjoint open sets separating σ from τ.

To see that the space is zerodimensional, notice that given $\tau \in U \subseteq \mathcal{G}_\omega$, the set $H_\tau = \{s \in \mathcal{G}_\omega : (\forall |\tau| \leq n \leq |s|)(s \restriction n \in U \;\&\; \tau \subseteq s)$ is a clopen subset of U containing s.

To see that it is ED consider an open set $U \subseteq \mathcal{G}_\omega$ with $t \in \overline{U}$. By recursion construct $\langle T_n : n < \omega \rangle$ such that $T_n \subseteq U$ and for each $s \in T_n$ the set $\{k : s^\frown k \in T_{n+1}\}$ is in p. Let $T_0 = \{t\}$. If we have constructed T_n and $s \in T_n$ then, $L_s = \{k : s^\frown k \in \overline{U}\} \in p$ (This is clear if $s \in U$ and if not, then for each $k \in \omega \setminus L_s$ there would be an open U_k containing $s^\frown k$ and disjoint from U. But then $\{s\} \cup \bigcup_{k \in \omega \setminus L_{n+1}} U_k$ would be a neibourghood of s disjoint from U contradicting $s \in \overline{U}$.). Now let $T_{n+1} = \{s^\frown k : s \in T_n, k \in L_s\}$. This finishes the recursive definition and finally let $V = \bigcup_{n<\omega} T_n$. Then $V \subseteq \overline{U}$ is an open neibourghood of t showing that $\overline{U}$ is open.

Finally we show $\mathcal{G}_\omega$ is $\aleph_0$-bounded. First notice that, since $\mathcal{G}$ is zerodimensional, $\beta\mathcal{G}_\omega \approx \mathrm{Ult}(\mathrm{Clop}(\mathcal{G}_\omega))$. We introduce some notation. For $s \in \mathcal{G}_\omega$ let $\mathcal{G}_\omega(s) = \{t \in \mathcal{G}_\omega : s \subseteq t\}$, $L_s(n) = \{t \in \mathcal{G}_\omega(s) : |t| = n + |s|\}$, $\mathrm{succ}(s) = \{t \in L_s(1)\}$ and, given an open $U \subseteq \mathcal{G}_\omega$ let $\hat{U} = \{q \in \beta\mathcal{G}_\omega : U \in q\}$. In the following, closure will always be taken in $\beta\mathcal{G}_\omega$ unless otherwise stated.

4.3 Observation. *Each $\widehat{\mathcal{G}_\omega(s)}$ is a clopen subset of $\beta\mathcal{G}_\omega$ disjoint from $\overline{(\beta\mathcal{G}_\omega \setminus \mathcal{G}_\omega(s))}$.*

Note that $\overline{\mathrm{succ}(s)}$ is isomorphic to $\beta\omega$ with s being taken to p by the isomorphism. Since p was a weak P-point, together with the above, we have:

4.4 Observation. *Each $s \in \mathcal{G}_\omega$ is a weak P-point in $\overline{L_\emptyset(|s|+1)}$.*

Let $D = \{p_n : n < \omega\}$ be a countable subset of $\beta\mathcal{G}_\omega \setminus \mathcal{G}_\omega$ and $t \in \mathcal{G}_\omega$. We will find a neibourghood U of t disjoint from D. Let $X_n = \overline{L_t(n)}$ and $D_n = D \cap X_n$. We shall recursively build a neibourghood T of t in $\mathcal{G}_\omega$ and in the end let $U = \hat{T} \cap \overline{\mathcal{G}_\omega(t)}$. We let $T_0 = \{t\}$ and suppose we have constructed T_n. Since each $s \in T_n$ is a weak P-point of $\overline{L_s(n+1)}$ and since X_{n+1} is countable and disjoint from $\mathcal{G}_\omega$, we may find an open set (in $\beta\mathcal{G}_\omega$) U_s such that $s \in U_s$ and $U_s \cap D_{n+1} = \emptyset$. Since $\beta\mathcal{G}_\omega \approx \mathrm{Ult}(\mathrm{Clop}(\mathcal{G}_\omega))$, we may find a clopen $U'_s \subseteq \mathcal{G}_\omega$ such that $\hat{U}'_s \subseteq U_s$. Let $A_s = \{U' \cap \mathrm{succ}(s)\}$ and let $T_{n+1} = T_n \cup \bigcup_{s \in T_n} A_s$. At the end of the recursion let $T = \bigcup_{n<\omega} T_n$ and $U = \hat{T} \cap \widehat{\mathcal{G}_\omega(t)}$. Then U is an open neibourghood of t disjoint from $\bigcup_{n<\omega} D_n$. Finally we let $D' = D \setminus \bigcup_{n<\omega} D_n$ and it remains to find a neibourghood of t disjoint from D'. Enumerate D' as $\{q_n : n < \omega\}$ and pick U_n a clopen neibourghood of $\{q_i : i \leq n\}$ disjoint from $\overline{L_n(t)}$ and let $U = \bigcup_{n<\omega} U_n$. We claim that $V = \mathcal{G}_\omega(t) \setminus U$ is an open neibourghood of t (in $\mathcal{G}_\omega$). Pick $s \in \mathcal{G}_\omega(t) \setminus U$ and suppose to the contrary that $\{n : s^\frown n \notin V\} \in p$. Then $s \in \overline{\{s^\frown n : s^\frown n \notin V\}} \subseteq \overline{U}$ and, by the definition of U, $s \in \overline{\bigcup_{i \leq |s|-|t|} U_i}$ but this is impossible, since $\bigcup_{i \leq |s|-|t|} U_i$ is clopen so then it would follow that $s \in \bigcup_{i \leq |s|-|t|} U_i \subseteq U$ a contradiction with the choice of s. □

Note that if a space X has an $\aleph_0$-bounded remainder, any finer topology also has an $\aleph_0$-bounded remainder:

4.5 Proposition. *If $(X,\tau)^*$ is a zerodimensional $\aleph_0$-bounded space and $\sigma \supseteq \tau$ is also zerodimensional, then $(X,\sigma)^*$ is $\aleph_0$ bounded.*

Proof. Note that any $p \in (X,\tau)^*$ corresponds to a closed subset of $(X,\sigma)^*$ (denote it $[p]$). Now given $\{q_n : n < \omega\} \subseteq (X,\sigma)^*$ we can find $\{p_n : n < \omega\} \subseteq (X,\tau)^*$ such that $\{q_n : n < \omega\} \subseteq \bigcup\{[p_n] : n < \omega\}$. Since $(X,\tau)^*$ is $\aleph_0$-bounded, $\overline{\{p_n : n < \omega\}}^{\beta(X,\tau)} \cap X = \emptyset$ so also $\overline{\{q_n : n < \omega\}}^{\beta(X,\sigma)} \cap X = \emptyset$ which implies that $(X,\sigma)^*$ is $\aleph_0$-bounded. □

4.6 THEOREM. *There is a countable, ED, perfectly disconnected space X with an $\aleph_0$-bounded remainder.*

Proof. Take the space $\mathcal{G}_\omega$ from theorem 4.2, and refine the topology to a maximal regular topology. Then, by the previous proposition, this space still has an $\aleph_0$-bounded remainder and so does its open perfectly disconnected subspace given by theorem 3.5. Let X be this subspace. □

4.7 THEOREM. *ω^* contains a lonely point.*

Proof. Let X be the space from the previous theorem. Since it is crowded perfectly disconnected, each of its points is a lonely point of X. Since its remainder is $\aleph_0$-bounded, each of its points is also a lonely point of βX. Since it is ED, βX is also ED and since it is countable, βX has weight at most $2^{\aleph_0}$. Hence, by theorem 2.5, βX can be embedded as a weak P-set into ω^* and each point of X will be a lonely point of ω^* (by observation 2.2). □

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