

FILTER CONVERGENCE IN $\beta\omega$

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ABSTRACT

We give a combinatorial condition which is necessary for a filter \mathcal{F} to admit injective \mathcal{F} -convergent sequences in $\beta\omega$. We also show that no analytic filter \mathcal{F} admits an injective \mathcal{F} -convergent sequence in $\beta\omega$. This answers a question of T. Banach, V. Mychaylyuk and L. Zdomskyy.

Keywords: analytic filter, meager filter, filter convergence, Rudin-Keisler ordering.

Introduction

It is well known that $\beta\omega$, the Čech-Stone compactification of the natural numbers, cannot contain convergent sequences.

On the other hand, if we generalize the notion of convergence to convergence with respect to a filter¹, we have the following easy observation.

Observation. *If X is a compact space and \mathcal{U} is an ultrafilter, then any sequence in X converges with respect to \mathcal{U} .*

Ultrafilters are very far from the Fréchet filter which corresponds to the standard convergence, so the above observation is not very surprising. What is surprising, however, is that one can get very close — in the sense of category — to having convergent sequences in every compact space. More precisely, the following (see [2]) is true:

Theorem (Banach, Mychaylyuk, Zdomskyy). *Any infinite compact space contains an injective sequence which converges with respect to some meager filter.*

In the cited paper the authors ask about the borderline between filters \mathcal{F} , which admit an injective \mathcal{F} -convergent sequence in $\beta\omega$ and those that don't. In particular, they ask, whether there is some analytic filter and an injective sequence in $\beta\omega$ which would converge with respect to this filter. Below we give a necessary condition for a filter \mathcal{F} to admit an injective \mathcal{F} -convergent sequence in $\beta\omega$. As a corollary we give a negative answer to the above mentioned question by showing that no analytic filter admits a convergent sequence in $\beta\omega$.

¹ Unfamiliar concepts used in this introduction will be defined below.

1. Definitions

In this section we recall some basic definitions and facts. We first start by generalizing the notion of a convergent sequence.

1.1 Definition. If $\bar{x} = \langle x_n : n < \omega \rangle \subseteq X$ is a sequence, we say that it is *injective*, provided $x_n \neq x_k$ for $n \neq k$. Given a filter \mathcal{F} on ω and $x \in X$ we write

$$x = \mathcal{F} - \lim \bar{x}$$

if for every neighbourhood U of x the set

$$\{n : x_n \in U\}$$

is an element of \mathcal{F} . We say that x is an \mathcal{F} -*limit* of the sequence \bar{x} . If some injective sequence $\bar{x} \subseteq X$ has an \mathcal{F} -limit in X , we say that \mathcal{F} *admits an \mathcal{F} -convergent sequence in X* .

The notion of a filter limit in the case when \mathcal{F} is an ultrafilter was introduced in [3] in the context of nonstandard analysis, the same notion for filters was considered in [6, 1] in the context of dynamical systems. See also [5].

1.2 Observation. If $x = \mathcal{F} - \lim \bar{x}$ and $A \subseteq \omega$ is a positive (with respect to \mathcal{F}) set of indices, i.e. $A \in \mathcal{F}^+ = \{X \subseteq \omega : \omega \setminus X \notin \mathcal{F}\}$, then $x \in \overline{\{x_n : n \in A\}}$.

We now turn to filters (and ultrafilters) on ω .

1.3 Definition. Recall, that a filter on ω can be identified with a subset of the Cantor space $\mathcal{P}(\omega)$. We thus say that a filter is *closed*, *meager*, F_σ or *analytic* if it is such under this identification.

1.4 Definition ([7, 4] for ultrafilters). Given filters \mathcal{F}, \mathcal{H} we say that \mathcal{F} is *Rudin-Keisler* above \mathcal{H} if there is a function $f : \omega \rightarrow \omega$ such that $\mathcal{H} = f_*(\mathcal{F}) = \{A : f^{-1}[A] \in \mathcal{F}\}$. In this case we write $\mathcal{F} \geq_{RK} \mathcal{H}$.

1.5 Proposition. No analytic filter is RK-above a nonprincipal ultrafilter.

Proof. First notice that if $f : \omega \rightarrow \omega$ is a function, then the function $\tilde{f} : 2^\omega \rightarrow 2^\omega$ given by $\tilde{f}(A) = f^{-1}[A]$ is continuous. Now suppose \mathcal{F} is analytic and that \mathcal{F} is RK-above some ultrafilter \mathcal{U} as witnessed by f . Then, by definition, $\mathcal{U} = \{A : \tilde{f}(A) \in \mathcal{F}\} = \tilde{f}^{-1}[\mathcal{F}]$ so \mathcal{U} is a continuous preimage of an analytic set and hence is also analytic. Since analytic sets have the Baire property (see e.g. [8], Theorem 29.5) while nonprincipal ultrafilters cannot have the Baire property (see e.g. [8], Exercise 8.50), this implies that \mathcal{U} is principle. \square

We will also need the following definition/fact about $\beta\omega$

1.6 Definition. The space $\beta\omega$ can be identified with the set of ultrafilters on natural numbers. The base for the topology consists of (clopen) sets of the form \hat{A} for $A \subseteq \omega$ where

$$\hat{A} = \{p \in \beta\omega : A \in p\}.$$

2. Main theorem

2.1 Theorem. *If \mathcal{F} is a filter such that for any partition $\langle A_n : n < \omega \rangle$ of ω into sets not in \mathcal{F} there is a $B \subseteq \omega$ such that both $\bigcup_{n \in B} A_n$ and $\bigcup_{n \notin B} A_n$ are \mathcal{F} -positive, then no injective $\tilde{x} \subseteq \beta\omega$ has an \mathcal{F} -limit.*

Proof. Suppose that for each partition A_n of ω into sets not in \mathcal{F} there is a $B \subseteq \omega$ so that both $\bigcup_{n \in B} A_n$ and $\bigcup_{n \notin B} A_n$ are \mathcal{F} -positive. Aiming towards a contradiction assume there is some \tilde{x} and $p = \mathcal{F}\text{-}\lim \tilde{x}$. Without loss of generality $p \notin \tilde{x}$. Recursively pick $y_n \in \tilde{x}$ and pairwise disjoint basic clopen neighbourhoods \hat{C}_n of y_n , with $C_n \subseteq \omega$, such that no \hat{C}_n contains p and together they cover the sequence \tilde{x} . Let $A_n = \{k : x_k \in \hat{C}_n\}$. Then $A_n \notin \mathcal{F}$, since $p \notin \hat{C}_n$. By assumption we can find $B \subseteq \omega$, such that both $\bigcup_{n \in B} A_n$ and $\bigcup_{n \notin B} A_n$ are \mathcal{F} -positive. Since p is an ultrafilter it contains either $\bigcup_{n \in B} C_n$ or $\bigcup_{n \notin B} C_n$. In the former case, this contradicts the fact that set $\bigcup_{n \notin B} A_n$ is \mathcal{F} -positive, in the latter case it contradicts the fact that the set $\bigcup_{n \in B} A_n$ is \mathcal{F} -positive. \square

2.2 Corollary. *No analytic filter \mathcal{F} admits an injective \mathcal{F} -convergent sequence in $\beta\omega$.*

Proof. Assume \mathcal{F} admits an injective \mathcal{F} -convergent sequence in $\beta\omega$. By theorem 2.1 there must be a partition $\langle A_n : n < \omega \rangle$ of ω into sets not in \mathcal{F} such that for each $B \subseteq \omega$ either $\bigcup_{n \in B} A_n \in \mathcal{F}$ or $\bigcup_{n \notin B} A_n \in \mathcal{F}$. Let $g(k) = \min\{n : k \in A_n\}$. Then g shows that \mathcal{F} is RK-above the ultrafilter $\{B : \bigcup_{n \in B} A_n \in \mathcal{F}\}$ so, by proposition 1.5, \mathcal{F} cannot be analytic. \square

Note. Note that the fact we are talking about $\beta\omega$ and not just any space without a nontrivial convergent sequence is crucial here. The following construction shows that there are spaces without nontrivial convergent sequences which contain \mathcal{F} -convergent sequences even for some $F_{\sigma\delta}$ filters.

2.3 Proposition. *If \mathcal{F} is a P -filter properly extending the Fréchet filter then there is a compact space X which contains an injective \mathcal{F} -convergent sequence but does not contain convergent sequences.*

Proof. Let $X_{\mathcal{F}}$ be the quotient space $\beta\omega/H$, for $H = (\bigcap_{F \in \mathcal{F}} \bar{F}^{\beta\omega}) \setminus \omega$ and let z be the point corresponding to H . Then $X_{\mathcal{F}}$ is a compact space. We first show that $X_{\mathcal{F}}$ contains no injective convergent sequences. Let $\langle x_n : n < \omega \rangle$ be an injective sequence in $X_{\mathcal{F}}$ and without loss of generality assume $x_n \neq z$. Notice that the sequence cannot converge to a point in $X_{\mathcal{F}} \setminus \{z\}$ (otherwise it would be a convergent sequence in $\beta\omega$). So, aiming towards a contradiction, assume $z = \lim_{n \rightarrow \infty} x_n$. Since $x_n \notin H$ we can choose $A_n \in \mathcal{F}$ such that $A_n \not\ni x_n$. Since \mathcal{F} is a P -filter, we can find a pseudointersection $A \in \mathcal{F}$, i.e. $A \subseteq^* A_n$ for all $n < \omega$. Then A is a neighbourhood of z which misses each x_n — a contradiction. So it remains to show that $z = \mathcal{F}\text{-}\lim n$ (recall that $\omega \subseteq X_{\mathcal{F}}$), but this is obvious. \square

Note. This construction only works for P -filters. Indeed, if $\langle A_n : n < \omega \rangle$ are infinite disjoint subsets of ω and \mathcal{F} is the filter generated by cofinite sets and sets of the form $\{B : (\forall^\infty n)(A_n \subseteq B)\}$, then in $X_{\mathcal{F}}$ any sequence $\langle x_n : n < \omega \rangle$ such that $A_n \in x_n$ will converge to z . This motivates the following question

2.4 Question. Let \mathcal{F} be a filter properly extending the Fréchet filter. Can we always find a compact space X containing an injective \mathcal{F} -convergent sequence but no convergent sequence?

After reading the proof of theorem 2 in [2] one might well be tempted to ask:

2.5 Question. If \mathcal{F} is RK-above a nonprincipal ultrafilter \mathcal{U} , does it admit an injective \mathcal{F} -convergent sequence in every compact space? Does it, at least, admit an injective \mathcal{F} -convergent sequence in $\beta\omega$.

We conjecture the answer to this question is yes however all our attempts at proving this turned out to lead nowhere. Note that since a separable space is embeddable in ω^* iff it is extremally disconnected, to give a positive answer to the second question is equivalent to building an ED topology on $\omega \cup \{p\}$ such that $p = \mathcal{F} - \lim n$ in this topology.

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